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Dilatation structures I. Fundamentals

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Abstract

A dilatation structure is a concept in between a group and a differential structure. In this article we study fundamental properties of dilatation structures on metric spaces. This is a part of a series of papers which show that such a structure allows to do non-commutative analysis, in the sense of differential calculus, on a large class of metric spaces, some of them fractals. We also describe a formal, universal calculus with binary decorated planar trees, which underlies any dilatation structure.

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1 Introduction

The purpose of this paper is to introduce dilatation structures on metric spaces. A dilatation structure is a concept in between a group and a differential structure. Any metric space (X, d) endowed with a dilatation structure has an associated tangent bundle. The tangent space at a point is a conical group, that is the tangent space has a group structure together with a one-parameter group of automorphisms. Conical groups generalize Carnot groups, i.e. nilpotent groups endowed with a graduation. Each dilatation structure leads to a non-commutative differential calculus on the metric space (X, d).

There are several important papers dedicated to the study of extra structures on a metric space which allows to do a reasonable analysis in such spaces, like Cheeger [6] or Margulis-Mostow [10], [11].

The constructions proposed in this paper first appeared in connection to problems in analysis on sub-riemannian manifolds. Parts of this article can be seen as a rigorous formulation of the considerations in the last section of Bellaïche [1].

A dilatation structure is simply a bundle of semigroups of (quasi-)contractions on the metric space (X, d), satisfying a number of axioms. The tangent bundle structure associated with a given dilatation structure on the metric space (X, d) is obtained by a passage to the limit procedure, starting from an algebraic structure which lives on the metric space.

With the help of the dilatation structure we construct a bundle (over the metric space) of (local) operations: to each $x \in X$ and parameter ε, for simplicity here $\varepsilon \in (0, +\infty)$, there is a natural non-associative operation

$$\Sigma_{\varepsilon}^{x} : U(x) \times U(x) \to U(x) \quad ,$$

where $U(x)$ is a neighbourhood of x. The non-associativity of this operation is controlled by the parameter ε. As ε goes to 0 the operation Σ_{ε}^{x} converges to a group operation on the tangent space of (X, d) at x.

Denote by δ_{ε}^{x} the dilatation based at $x \in X$, of parameter ε. The bundle of operations satisfies a kind of weak associativity, even if for any fixed $y \in X$ the operation Σ_{ε}^{y} is non-associative. The weak associativity property, named also shifted associativity, is:

$$\Sigma_{\varepsilon}^{x}(u, \Sigma_{\varepsilon}^{\delta_{\varepsilon}^{x}}(v, w)) = \Sigma_{\varepsilon}^{x}(\Sigma_{\varepsilon}^{x}(u, v), w) \quad ,$$

for any $x \in X$ and any $u, v, w \in X$ sufficiently close to X. We shall describe also other objects (like a function satisfying a shifted inverse property) and algebraic identities related to the dilatation structure and the induced bundle of operations.

We briefly describe further the contents of the paper. In section 2 we give motivational examples of dilatation structures. Basic notions and results of metric geometry and groups endowed with dilatations are mentioned in section 3.

In section 4 we introduce a formalism based on decorated planar binary trees. This formalism will be used to prove the main results of the paper. We show that, from an algebraic point of view, dilatation structures (more precisely the formalism in section 4) induce a bundle of one parameter deformations of binary operations, which are not associative, but shifted associative. This is a structure which bears resemblance with the tangent bundle of a Lie group, but it is more general.

Section 5, 6 and 7 are devoted to dilatation structures. These sections contain the main results of the paper. After we introduce and explain the axioms of dilatation structures, we describe several key metric properties of such a structure, in section 5. In section 6 we prove that a dilatation structure induces a valid notion of tangent bundle. In section 7 we explain how a dilatation structure leads to a differential calculus.

Section 8 is made of two parts. In the first part we show that dilatation structures induce differential structures, in a generalized sense. In the second part we turn to conical groups and we prove the curious result that, even if in a conical group left translations are smooth but right translations are generically non differentiable, the group operation is smooth if we well choose a dilatation structure.

2 Motivation

We start with a trivial example of a dilatation structure, then we briefly explain the occurence of such a structure in more unusual situations.

There is a lot of structure hiding in the dilatations of $\mathbb{R}^n$. For this space, the dilatation based at x, of coefficient $\varepsilon > 0$, is the function

$$\delta_\varepsilon^x : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad , \quad \delta_\varepsilon^x y = x + \varepsilon(-x+y) \quad .$$

For fixed x the dilatations based at x form a one parameter group which contracts any bounded neighbourhood of x to a point, uniformly with respect to x.

Dilatations behave well with respect to the euclidean distance d, in the following sense: for any $x, u, v \in \mathbb{R}^n$ and any $\varepsilon > 0$ we have

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) = d(u,v) \quad .$$

This shows that from the metric point of view the space $(\mathbb{R}^n, d)$ is a metric cone, that is $(\mathbb{R}^n, d)$ looks the same at all scales.

Moreover, let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function and $x \in \mathbb{R}^n$. The function f is differentiable in x if there is a linear transformation A (that is a group morphism which commutes with dilatations based at the neutral element 0) such that the limit

$$\lim_{\varepsilon \rightarrow 0} \delta_{\varepsilon^{-1}}^{f(x)} f \delta_\varepsilon^x (v) = f(x) + A(-x+v) \tag{2.1}$$

is uniform with respect to v in bounded neighbourhood of x. Indeed, let us compute:

$$\delta_{\varepsilon^{-1}}^{f(x)} f \delta_\varepsilon^x (v) = f(x) + \frac{1}{\varepsilon}(-f(x) + f(x + \varepsilon(-x+v))) \quad .$$

This shows that we get the usual definition of differentiability.

The relation (2.1) can be put in another form, using the euclidean distance:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} d(\delta_\varepsilon^{f(x)} T(x)(v), f(\delta_\varepsilon^x v)) = 0$$

uniformly with respect to v in bounded neighbourhood of x. Here $T(x)(v) = x + A(-x+v)$. In conclusion, dilatations are the fundamental object for doing differential calculus on $\mathbb{R}^n$.

Even the algebraic structure of $\mathbb{R}^n$ is encoded in dilatations. Indeed, we can recover the operation of addition from dilatations. It goes like this: for $x, u, v \in \mathbb{R}^n$ and $\varepsilon > 0$ define

$$\Delta_\varepsilon^x(u,v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \quad , \quad \Sigma_\varepsilon^x(u,v) = \delta_{\varepsilon^{-1}}^x \delta_\varepsilon^{\delta_\varepsilon^x u}(v) \quad , \quad inv_\varepsilon^x(u) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} x \quad .$$

For fixed x, u, ε the functions $\Delta_\varepsilon^x(u,\cdot), \Sigma_\varepsilon^x(u,\cdot)$ are inverse one to another, but we don't insist on this for the moment (see proposition 3).

What is the meaning of these functions? Let's compute:

$$\Delta_\varepsilon^x(u,v) = \delta_\varepsilon^x u + \frac{1}{\varepsilon}\left(-\left(\delta_\varepsilon^x u\right) + \delta_\varepsilon^x v\right) =$$

$$= (x + \varepsilon(-x+u)) + \frac{1}{\varepsilon}\left(\varepsilon(-u+x) - x + x + \varepsilon(-x+v)\right) =$$

$$= x + \varepsilon(-x+u) + \frac{1}{\varepsilon}\varepsilon(-u+v) =$$

$$= x + \varepsilon(-x+u) + (-u+v) \quad ,$$

$$\Sigma^x_\varepsilon(u,v) = x + \frac{1}{\varepsilon}\left(-x + \delta^x_\varepsilon u + \varepsilon\left(-\left(\delta^x_\varepsilon u\right) + v\right)\right) =$$

$$= x + \frac{1}{\varepsilon}\left(\varepsilon(-x+u) + \varepsilon\left(\varepsilon(-u+x) - x + v\right)\right) =$$

$$= u + \varepsilon(-u+x) + (-x+v) \quad .$$

In the same way we get

$$inv^x_\varepsilon(u) == x + \varepsilon(-x+u) + (-u+x) \quad .$$

As $\varepsilon \to 0$ we have the limits:

$$\lim_{\varepsilon\to 0} \Delta^x_\varepsilon(u,v) = \Delta^x(u,v) = x + (-u+v) \quad ,$$

$$\lim_{\varepsilon\to 0} \Sigma^x_\varepsilon(u,v) = \Sigma^x(u,v) = u + (-x+v) \quad ,$$

$$\lim_{\varepsilon\to 0} inv^x_\varepsilon(u) = inv^x(u) = x - u + x \quad ,$$

uniform with respect to x, u, v in bounded sets. The function $\Sigma^x(\cdot,\cdot)$ is a group operation, namely the addition operation translated such that the neutral element is x. Thus, for $x = 0$, we recover the group operation. The function $inv^x(\cdot)$ is the inverse function, and $\Delta^x(\cdot,\cdot)$ is the difference function.

Notice that for fixed x, ε the function $\Sigma^x_\varepsilon(\cdot,\cdot)$ is not a group operation, first of all because it is not associative. Nevertheless, this function satisfies a shifted associativity property, namely (see proposition 5)

$$\Sigma^x_\varepsilon(\Sigma^x_\varepsilon(u,v),w) = \Sigma^x_\varepsilon(u, \Sigma^{\delta^x_\varepsilon u}_\varepsilon(v,w)) \quad .$$

Also, the inverse function inv^x_ε is not involutive, but shifted involutive (proposition 4):

$$inv^{\delta^x_\varepsilon u}_\varepsilon\left(inv^x_\varepsilon u\right) = u \quad .$$

These and other properties of dilatations allow to recover the structure of the tangent bundle of $\mathbb{R}^n$, which is trivial in this case.

Let us go to more elaborate examples. We may look to a riemannian manifold M, which is locally a deformation of $\mathbb{R}^n$. We can use charts for transporting (locally) the dilatation structure from $\mathbb{R}^n$ to the manifold. All the previously described metric and algebraic properties will hold in this situation, in a weaker form. For example the riemannian distance is no longer scalling invariant, but we still have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) = d^x(u,v) \quad ,$$

uniform limit with respect tu x, u, v in (small) bounded sets. Here d^x is an euclidean distance which can be identified with the distance in the tangent space of M at x, induced by the metric tensor at x. In the same way we can construct the algebraic structure of the tangent space at x, using the functions $\Sigma^x_\varepsilon, \Delta^x_\varepsilon$. We will have a differentiability notion coming from the dilatations transported by the chart.

If we change charts or the riemannian metric then the dilatation structure will change too, but not very much, essentially because the change of charts is smooth, therefore we are still able to say what are tangent spaces and to describe their algebraic structure.

Let us go further with more complex examples. Consider the Heisenberg group $H(n)$. AS a set $H(1) = \mathbb{R}^{2n} \times \mathbb{R}$. We shall use the following notation: an element of $H(n)$ will be denoted by $\tilde{x} = (x, \bar{x})$, with $x \in \mathbb{R}^{2n}, \bar{x} \in \mathbb{R}$. The group operation is

$$\tilde{x}\tilde{y} = (x + y, \bar{x} + \bar{y} + 2\omega(x, y)) \quad ,$$

where ω is the canonic symplectic 2-form on $\mathbb{R}^{2n}$.

The group $H(n)$ is nilpotent, in fact a 2 graded Carnot group. This means that $H(n)$ is nilpotent and that it admits a one-parameter group of isomorphisms

$$\delta_\varepsilon(x, \bar{x}) = (\varepsilon x, \varepsilon^2 \bar{x}) \quad .$$

These are dilatations, more precisely we can construct dilatations based at $\tilde{x}$ by the formula

$$\delta_\varepsilon^{\tilde{x}} \tilde{u} = \tilde{x} \delta_\varepsilon \left(\tilde{x}^{-1} \tilde{u} \right) \quad .$$

We may also put a scalling invariant distance on $H(n)$, for example this one:

$$d(\tilde{x}, \tilde{y}) = g(\tilde{x}^{-1}\tilde{y}) \quad , \quad g(\tilde{u}) = \max \left\{ \|u\|, \sqrt{\mid \bar{u} \mid} \right\} \quad .$$

We can repeat step by step the constructions explained before in this situation. There are some differences though.

First of all, from the metric point of view, $(H(n), d)$ is a fractal space, in the sense that the Hausdorff dimension of this space is equal to $2n+2$, therefore strictly greater than the topological dimension, which is $2n+1$. Second, the differential of a function defined by the dilatations is not the usual differential, but an essentially different one, called Pansu derivative (see [13]). This is part of a very active area of research in geometric analysis (among fundamental references one may cite [13], [6], [10], [11], [7]). A spectacular application of Pansu derivative was to prove a Rademacher theorem which in turn implies deep results about Mostow rigidity. The theory applies to general Carnot groups.

The Heisenberg group is not commutative. It is in fact the model for the tangent space of a contact metric manifold, as the euclidean $\mathbb{R}^n$ is the model of the tangent space of a riemannian manifold. We enter here in the realm of sub-riemannian geometry (see for example [1], [9]). In a future paper we shall deal with dilatation structures for sub-riemannian manifolds. An important problem in sub-riemannian geometry is to have good tangent bundle structures, which in turn allow us to prove basic theorems, like Poincaré inequality, Rademacher or Stepanov theorem.

We may even go further and find dilatation structures related with rectifiable sets, or with some self-similar sets. This is not the purpose of this paper though. In the sequel we shall define and study fundamental properties of dilatation structures.

3 Basic notions

We denote by $f \subset X \times Z$ a relation and we write $f(x) = y$ if $(x, y) \in f$. Therefore we may have $f(x) = y$ and $f(x) = y'$ with $y \neq y'$, if $(x, y) \in f$ and $(x, y') \in f$.

The domain of f is the set of $x \in X$ such that there is $z \in Z$ with $f(x) = z$. We denote the domain by *dom* f. The image of f is the set of $z \in Z$ such that there is $x \in X$ with $f(x) = z$. We denote the image by *im* f.

By convention, when we state that a relation $R(f(x), f(y), ...)$ is true, it means that $R(x', y', ...)$ is true for any choice of $x', y', ...$, such that $(x, x'), (y, y'), ... \in f$.

In a metric space (X, d), the ball centered at $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. If we need to emphasize the dependence on the distance d then we shall use the notation $B_d(x, r)$. In the same way, $\bar{B}(x, r)$ and $\bar{B}_d(x, r)$ denote the closed ball centered at x, with radius r.

We shall use the following convenient notation: by $\mathcal{O}(\varepsilon)$ we mean a positive function such that $\lim_{\varepsilon \to 0} \mathcal{O}(\varepsilon) = 0$.

3.1 Gromov-Hausdorff distance

There are several definitions of distances between metric spaces. For this subject see Burago & al. [5] section 7.4, Gromov [8], chapter 3, Gromov [7].

We explain now a well-known alternative definition of the Gromov-Hausdorff distance, up to a multiplicative factor.

Definition 1. Let (X_i, d_i, x_i), $i = 1, 2$, be a pair of locally compact pointed metric spaces and $\mu > 0$. We shall say that μ is admissible if there is a relation $\rho \subset X_1 \times X_2$ such that:

1. *dom* ρ is μ-dense in X_1,
2. *im* ρ is μ-dense in X_2,
3. $(x_1, x_2) \in \rho$,

4 for all $x, y \in$ *dom* ρ we have :

$$\mid d_2(\rho(x), \rho(y)) - d_1(x, y) \mid \leq \ \mu \ . \tag{3.1}$$

The Gromov-Hausdorff distance between (X_1, x_1, d_1) and (X_2, x_2, d_2) is the infimum of admissible numbers μ.

Denote by $[X, d_X, x]$ the isometry class of (X, d_X, x), that is the class of spaces (Y, d_Y, y) such that it exists an isometry $f : X \to Y$ with the property $f(x) = y$. Note that if (X, d_X, x) is isometric with (Y, d_Y, y) then they have the same diameter.

The Gromov-Hausdorff distance is in fact almost a distance between isometry classes of pointed metric spaces. Indeed, if two pointed metric spaces are isometric then the Gromov-Hausdorff distance equals 0. The converse is also true in the class of compact (pointed) metric spaces (Gromov [8] proposition 3.6).

Moreover, if two of the isometry classes $[X, d_X, x]$, $[Y, d_Y, y]$, $[Z, d_Z, z]$ have (representants with) diameter at most equal to 2, then the triangle inequality is true. We shall use this distance and the induced convergence for isometry classes of the form $[X, d_X, x]$, with *diam* $X \ \leq 3/2$.

3.2 Metric profiles. Metric tangent space

We shall denote by CMS the set of isometry classes of pointed compact metric spaces. The distance on this set is the Gromov distance between (isometry classes of) pointed metric spaces and the topology is induced by this distance.

To any locally compact metric space we can associate a metric profile (Buliga [3], [4]).

Definition 2. The metric profile associated to the locally metric space (M, d) is the assignment (for small enough $\varepsilon > 0$)

$$(\varepsilon > 0, \ x \in M) \ \mapsto \ \mathbb{P}^m(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon} d, x] \in CMS$$

We can define a notion of metric profile regardless to any distance.

Definition 3. A metric profile is a curve $\mathbb{P} : [0, a] \to CMS$ such that:

(a) it is continuous at 0,

(b) for any $b \in [0, a]$ and $\varepsilon \in (0, 1]$ we have

$$d_{GH}(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x_b)) \;=\; O(\varepsilon)$$

(The function $\mathcal{O}(\varepsilon)$ may change with b.)

We used the notations:

$$\mathbb{P}(b) = [\bar{B}(x, 1), d_b, x_b] \quad \text{and} \quad \mathbb{P}^m_{d_b}(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon} d_b, x_b] \quad .$$

The metric profile is nice if

$$d_{GH}(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x)) \;=\; O(b\varepsilon).$$

Imagine that $1/b$ represents the magnification on the scale of a microscope. We use the microscope to study a specimen. For each $b > 0$ the information that we get is the table of distances of the pointed metric space $(\bar{B}(x, 1), d_b, x_b)$.

How can we know, just from the information given by the microscope, that the string of "images" that we have corresponds to a real specimen? The answer is that a reasonable check is the relation from point (b) of the definition of metric profiles 3.

Indeed, this point says that starting from any magnification $1/b$, if we further select the ball $\bar{B}(x, \varepsilon)$ in the snapshot $(\bar{B}(x, 1), d_b, x_b)$, then the metric space $(\bar{B}(x, 1), \frac{1}{\varepsilon} d_b, x_b)$ looks approximately the same as the snapshot $(\bar{B}(x, 1), d_{b\varepsilon}, x_b)$. That is: further magnification by ε of the snapshot (taken with magnification) b is roughly the same as the snapshot $b\varepsilon$. This is of course true in a neighbourhood of the base point x_b.

The point (a) from the definition 3 has no other justification than proposition 1 in next subsection.

We rewrite definition 1 with more details, in order to clearly understand what is a metric profile. For any $b \in (0, a]$ and for any $\mu > 0$ there is $\varepsilon(\mu, b) \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon(\mu, b))$ there exists a relation $\rho = \rho_{\varepsilon,b} \subset \bar{B}_{d_b}(x_b, \varepsilon) \times \bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$ such that:

1. *dom* $\rho_{\varepsilon,b}$ is μ-dense in $\bar{B}_{d_b}(x_b, \varepsilon)$,

2. *im* $\rho_{\varepsilon,b}$ is μ-dense in $\bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$,

3. $(x_b, x_{b\varepsilon}) \in \rho_{\varepsilon,b}$,

4. for all $x, y \in$ *dom* $\rho_{\varepsilon,b}$ we have :

$$\mid \frac{1}{\varepsilon} d_b(x, y) - d_{b\varepsilon}(\rho_{\varepsilon,b}(x), \rho_{\varepsilon,b}(y)) \mid \leq \; \mu \; . \tag{3.2}$$

In the microscope interpretation, if $(x, u) \in \rho_{\varepsilon,b}$ means that x and u represent the same "real" point in the specimen.

Therefore a metric profile gives two types of information:

- a distance estimate like (3.2) from point 4,

- an "approximate shape" estimate, like in the points 1–3, where we see that two sets, namely the balls $\bar{B}_{d_b}(x_b, \varepsilon)$ and $\bar{B}_{d_{b\varepsilon}}(x_{b\varepsilon}, 1)$, are approximately isometric.

The simplest metric profile is one with $(\bar{B}(x_b, 1), d_b, x_b) = (X, d_b, x)$. In this case we see that $\rho_{\varepsilon,b}$ is approximately an ε dilatation with base point x.

This observation leads us to a particular class of (pointed) metric spaces, namely the metric cones.

Definition 4. A metric cone (X, d, x) is a locally compact metric space (X, d), with a marked point $x \in X$ such that for any $a, b \in (0, 1]$ we have

$$\mathbb{P}^m(a, x) = \mathbb{P}^m(b, x).$$

Metric cones have dilatations. By this we mean the following:

Definition 5. Let (X, d, x) be a metric cone. For any $\varepsilon \in (0, 1]$ a dilatation is a function $\delta^x_\varepsilon : \bar{B}(x, 1) \to \bar{B}(x, \varepsilon)$ such that:

- $\delta^x_\varepsilon(x) \ = \ x$,
- for any $u, v \in X$ we have
$$d(\delta^x_\varepsilon(u), \delta^x_\varepsilon(v)) \ = \ \varepsilon \ d(u, v) \ .$$

The existence of dilatations for metric cones comes from the definition 4. Indeed, dilatations are just isometries from $(\bar{B}(x, 1), d, x)$ to $(\bar{B}, \frac{1}{a} d, x)$.

Metric cones are good candidates for being tangent spaces in the metric sense.

Definition 6. A (locally compact) metric space (M, d) admits a (metric) tangent space in $x \in M$ if the associated metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ (as in definition 2) admits a prolongation by continuity in $\varepsilon = 0$, i.e. if the following limit exists:

$$[T_x M, d^x, x] \ = \ \lim_{\varepsilon \to 0} \mathbb{P}^m(\varepsilon, x) \ . \tag{3.3}$$

The connection between metric cones, tangent spaces and metric profiles in the abstract sense is made by the following proposition.

Proposition 1. *The associated metric profile $\varepsilon \mapsto \mathbb{P}^m(\varepsilon, x)$ of a metric space (M, d) for a fixed $x \in M$ is a metric profile in the sense of the definition 3 if and only if the space (M, d) admits a tangent space in x.*

In such a case the tangent space is a metric cone.

Proof. Indeed, a tangent space $[V, d_v, v]$ exists if and only if we have the limit from the relation (3.3). In this case there exists a prolongation by continuity to $\varepsilon = 0$ of the metric profile $\mathbb{P}^m(\cdot, x)$. The prolongation is a metric profile in the sense of definition 3. Indeed, we have still to check the property (b). But this is trivial, because for any $\varepsilon, b > 0$, sufficiently small, we have

$$\mathbb{P}^m(\varepsilon b, x) \ = \ \mathbb{P}^m_{d_b}(\varepsilon, x)$$

where $d_b \ = \ (1/b)d$ and $\mathbb{P}^m_{d_b}(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon} d_b, x]$.

Finally, let us prove that the tangent space is a metric cone. Indeed, for any $a \in (0, 1]$ we have

$$[\bar{B}(x, 1), \frac{1}{a} d^x, x] \ = \ \lim_{\varepsilon \to 0} \mathbb{P}^m(a\varepsilon, x) \ .$$

Therefore $[\bar{B}(x, 1), \dfrac{1}{a} d^x, x] \ = \ [T_x M, d^x, x]$. □

3.3 Groups with dilatations. Virtual tangent space

In section 6 we shall see that metric tangent spaces sometimes have a group structure which is compatible with dilatations. This structure, of a group with dilatations, is interesting by itself. The notion has been introduced in [2]; we describe it further.

We start with the following setting: G is a topological group endowed with an uniformity such that the operation is uniformly continuous. The description that follows is slightly non canonical, but is nevertheless motivated by the case of a Lie group endowed with a Carnot-Caratheodory distance induced by a left invariant distribution.

We introduce first the double of G, as the group $G^{(2)} = G \times G$ with operation

$$(x,u)(y,v) = (xy, y^{-1}uyv) \quad .$$

The operation on the group G, seen as the function

$$op : G^{(2)} \rightarrow G \ , \quad op(x,y) = xy \quad ,$$

is a group morphism. Also the inclusions:

$$i' : G \rightarrow G^{(2)} \ , \quad i'(x) = (x,e)$$

$$i" : G \rightarrow G^{(2)} \ , \quad i"(x) = (x, x^{-1})$$

are group morphisms.

Definition 7. 1. G is an uniform group if we have two uniformity structures, on G and G^2, such that op, i', $i"$ are uniformly continuous.

2. A local action of a uniform group G on a uniform pointed space (X, x_0) is a function $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_\phi \in \mathcal{V}(x_0) \rightarrow V_\phi \in \mathcal{V}(x_0)$ such that:
   (a) the map $(\phi, x) \mapsto \hat{\phi}(x)$ is uniformly continuous from $G \times X$ (with product uniformity) to X,
   (b) for any $\phi, \psi \in G$ there is $D \in \mathcal{V}(x_0)$ such that for any $x \in D$ $\widehat{\phi\psi^{-1}}(x)$ and $\hat{\phi}(\hat{\psi}^{-1}(x))$ make sense and $\widehat{\phi\psi^{-1}}(x) = \hat{\phi}(\hat{\psi}^{-1}(x))$.

3. Finally, a local group is an uniform space G with an operation defined in a neighbourhood of $(e,e) \subset G \times G$ which satisfies the uniform group axioms locally.

Remark that a local group acts locally at left (and also by conjugation) on itself.

This definition deserves an explanation. An uniform group, according to the definition (7), is a group G such that left translations are uniformly continuous functions and the left action of G on itself is uniformly continuous too. In order to precisely formulate this we need two uniformities: one on G and another on $G \times G$.

These uniformities should be compatible, which is achieved by saying that i', $i"$ are uniformly continuous. The uniformity of the group operation is achieved by saying that the op morphism is uniformly continuous.

Definition 8. A group with dilatations (G, δ) is a local uniform group G with a local action of Γ (denoted by δ), on G such that

H0. the limit $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon x = e$ exists and is uniform with respect to x in a compact neighbourhood of the identity e.

H1. the limit

$$\beta(x,y) \;=\; \lim_{\varepsilon\to 0} \delta_\varepsilon^{-1}\left((\delta_\varepsilon x)(\delta_\varepsilon y)\right)$$

is well defined in a compact neighbourhood of e and the limit is uniform.

H2. the following relation holds

$$\lim_{\varepsilon\to 0} \delta_\varepsilon^{-1}\left((\delta_\varepsilon x)^{-1}\right) \;=\; x^{-1}$$

where the limit from the left hand side exists in a neighbourhood of e and is uniform with respect to x.

These axioms are the prototype of a dilatation structure.

The "infinitesimal version" of an uniform group is a conical local uniform group.

Definition 9. A conical group N is a local group with a local action of $(0,+\infty)$ by morphisms δ_ε such that $\lim_{\varepsilon\to 0} \delta_\varepsilon x \;=\; e$ for any x in a neighbourhood of the neutral element e.

Here comes a proposition which explains why a conical group is the infinitesimal version of a group with dilatations.

Proposition 2. *Under the hypotheses H0, H1, H2 (G,β,δ) is a conical group, with operation β and dilatations δ.*

Proof. All the uniformity assumptions allow us to change at will the order of taking limits. We shall not insist on this further and we shall concentrate on the algebraic aspects.

We have to prove the associativity, existence of neutral element, existence of inverse and the property of being conical.

For the associativity $\beta(x,\beta(y,z)) = \beta(\beta(x,y),z)$ we compute:

$$\beta(x,\beta(y,z)) = \lim_{\varepsilon\to 0,\eta\to 0} \delta_\varepsilon^{-1}\left\{(\delta_\varepsilon x)\delta_{\varepsilon/\eta}\left((\delta_\eta y)(\delta_\eta z)\right)\right\} \quad .$$

We take $\varepsilon = \eta$ and we get

$$\beta(x,\beta(y,z)) = \lim_{\varepsilon\to 0} \{(\delta_\varepsilon x)(\delta_\varepsilon y)(\delta_\varepsilon z)\} \quad .$$

In the same way:

$$\beta(\beta(x,y),z) = \lim_{\varepsilon\to 0,\eta\to 0} \delta_\varepsilon^{-1}\left\{(\delta_{\varepsilon/\eta} x)\left((\delta_\eta x)(\delta_\eta y)\right)(\delta_\varepsilon z)\right\} \quad .$$

and again taking $\varepsilon = \eta$ we obtain

$$\beta(\beta(x,y),z) = \lim_{\varepsilon\to 0} \{(\delta_\varepsilon x)(\delta_\varepsilon y)(\delta_\varepsilon z)\} = \beta(x,\beta(y,z)) \quad .$$

The neutral element is e, from H0 (first part): $\beta(x,e) = \beta(e,x) = x$. The inverse of x is x^{-1}, by a similar argument:

$$\beta(x,x^{-1}) = \lim_{\varepsilon\to 0,\eta\to 0} \delta_\varepsilon^{-1}\left\{(\delta_\varepsilon x)\left(\delta_{\varepsilon/\eta}(\delta_\eta x)^{-1}\right)\right\} \quad ,$$

and taking $\varepsilon = \eta$ we obtain

$$\beta(x,x^{-1}) = \lim_{\varepsilon\to 0} \delta_\varepsilon^{-1}\left((\delta_\varepsilon x)(\delta_\varepsilon x)^{-1}\right) = \lim_{\varepsilon\to 0} \delta_\varepsilon^{-1}(e) = \; e \quad .$$

Finally, β has the property:

$$\beta(\delta_\eta x, \delta_\eta y) = \delta_\eta \beta(x,y) \quad ,$$

which comes from the definition of β and commutativity of multiplication in $(0,+\infty)$. This proves that (G,β,δ) is conical. $\square$

In a sense (G, β, δ) is the tangent space of the group with dilatations (G, δ) at e. We can act with the conical group (G, β, δ) on (G, δ). Indeed, let us denote by $[f, g] = f \circ g \circ f^{-1} \circ g^{-1}$ the commutator of two transformations. For the group G we shall denote by $L^G_x y = xy$ the left translation and by $L^N_x y = \beta(x, y)$. The preceding proposition tells us that (G, β, δ) acts locally by left translations on G. We shall call the left translations with respect to the group operation β "infinitesimal". These infinitesimal translations admit an interesting commutator representation:

$$\lim_{\lambda \to 0} [L^G_{(\delta_\lambda x)^{-1}}, \delta^{-1}_\lambda] = L^N_x \quad . \tag{3.4}$$

Definition 10. The group VT_eG formed by all transformations L^N_x is called the virtual tangent space at e to G.

As local groups, VT_eG and (G, β, δ) are isomorphic. We can easily define dilatations on VT_eG, by conjugation with dilatations δ_ε. Indeed, we see that

$$L^N_{\delta_\varepsilon x}(y) = \beta(\delta_\varepsilon x, y) = \delta_\varepsilon L^N_x \, (\delta_\varepsilon)^{-1} \, .$$

The virtual tangent space VT_xG at $x \in G$ to G is obtained by translating the group operation and the dilatations from e to x. This means: define a new operation on G by

$$y \overset{x}{\cdot} z = yx^{-1}z \quad .$$

The group G with this operation is isomorphic to G with old operation and the left translation $L^G_x y = xy$ is the isomorphism. The neutral element is x. Introduce also the dilatations based at x by

$$\delta^x_\varepsilon y = x\delta_\varepsilon(x^{-1}y) \quad .$$

Then $G^x = (G, \overset{x}{\cdot})$ with the group of dilatations δ^x_ε satisfy the axioms H0, H1, H2. Define then the virtual tangent space VT_xG to be: $VT_xG = VT_xG^x$.

4 Binary decorated trees and dilatations

We want to explore what happens when we make compositions of dilatations (which depends also on $\varepsilon > 0$). The ε variable apart, any dilatation $\delta^x_\varepsilon(y)$ is a function of two arguments: x and y, invertible with respect to the second argument. The functions we can obtain when composing dilatations are difficult to write, that is why we shall use a tree notation.

4.1 The formalism

Let X be a non empty set and $\mathcal{T}(X)$ be a class of binary planar trees with leaves in X and all nodes decorated with two colors $\{\circ, \bullet\}$. The empty tree, that is the tree with no nodes or leaves, belongs to $\mathcal{T}(X)$. For any $x \in X$ we accept that there is a tree in $\mathcal{T}(X)$ with no nodes and with x as the only leaf. That is $X \subset \mathcal{T}(X)$.

For any color $\mathbf{a} \in \{\circ, \bullet\}$, let $\bar{\mathbf{a}}$ be the opposite color. The colors $\circ$ and $\bullet$ are codes for the symbols ε and ε^{-1}.

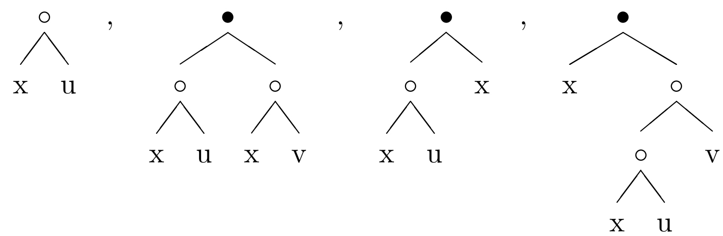
The relation "$\approx$" is an equivalence relation on $\mathcal{T}(X)$, taken as a primitive notion for the axioms which will follow.

The equivalence class of of a tree $\mathcal{P} \in \mathcal{T}(X)$ is denoted by $\begin{array}{c} \\ | \\ \mathcal{P} \end{array}$.

In various diagrams that will follow we shall use the notation $\begin{array}{c}\Gamma\\ |\\ \mathcal{P}\end{array}$ for saying that Γ is the equivalence class of $\mathcal{P}$.

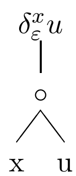
For any $\mathcal{P}, \mathcal{R} \in \mathcal{T}(X)$, " $\mathcal{P} \approx \mathcal{R}$" or " $\begin{array}{ccc} & = & \\ | & & | \\ \mathcal{P} & & \mathcal{R}\end{array}$ " means the same thing.

Axiom T0: *For any $x, u, v \in X$ the trees*

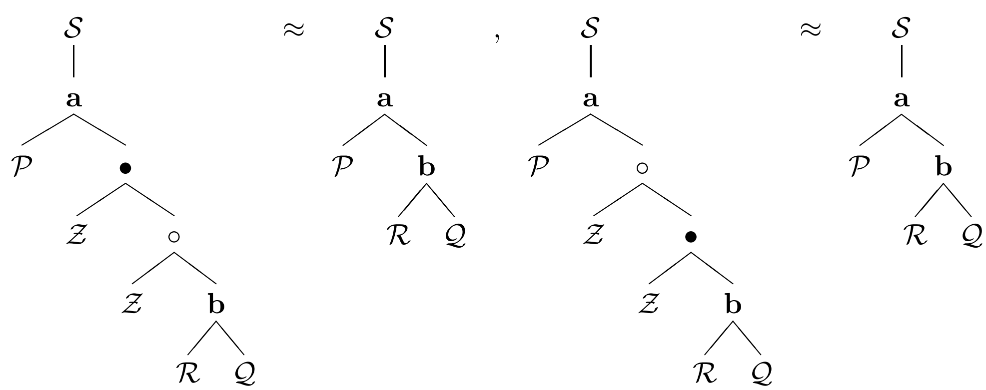


belong to $\mathcal{T}(X)$.

The equivalence class of $\underset{x \quad u}{\overset{\circ}{\wedge}}$ is denoted by $\delta^x_\varepsilon u$, that is we have:



Axiom T1: *Consider any trees $\mathcal{P}, \mathcal{R}, \mathcal{S}, \mathcal{Q}, \mathcal{Z} \in \mathcal{T}(X)$, any $x \in X$, and any colors* $\mathbf{a}, \mathbf{b}$ *such that the trees from the right hand sides of relations below belong to $\mathcal{T}(X)$. Then the trees from the left hand sides of relations below belong to $\mathcal{T}(X)$ and we have:*

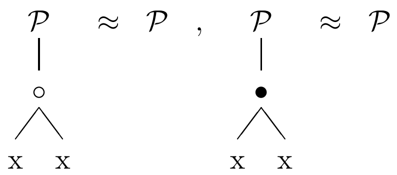

(4.1)

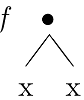
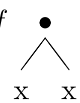
Here, in all diagrams, the symbol $\begin{array}{c}\mathcal{S}\\ |\\ \mathbf{a}\end{array}$ *means that the node colored with* $\mathbf{a}$ *is grafted at an arbitrary leaf of the tree $\mathcal{S}$.*

The second axiom expresses the fact that the dilatation (of any coefficient ε) δ^x_ε has x as fixed point, that is $\delta^x_\varepsilon x = x$.

Axiom T2: *For any $x \in X$ the tree* $\underset{x \quad x}{\overset{\bullet}{\wedge}}$ *belongs to $\mathcal{T}(X)$. Moreover, consider any tree $\mathcal{P} \in \mathcal{T}(X)$ and any $x \in X$. Then the trees from the left hand sides of relations below belong to*

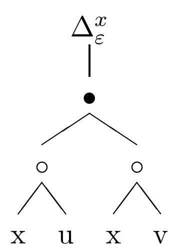
$\mathcal{T}(X)$ and we have:


(4.2)

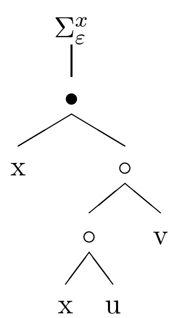
that is the equivalence class of x is the same as the equivalence class of  *and the equivalence class of*  *. As in axiom T1, the symbol*  *means that the root of the tree $\mathcal{P}$ is grafted at an arbitrary leaf of the tree $\mathcal{S}$.*

Definition 11. We define the difference, sum and inverse trees by:

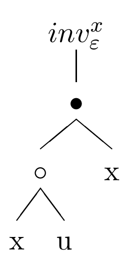
(a) the difference tree $\Delta^x_\varepsilon = \Delta^x_\varepsilon(u, v)$ given by relation



(b) the sum tree $\Sigma^x_\varepsilon = \Sigma^x_\varepsilon(u, v)$ given by relation



(c) the inverse tree $inv^x_\varepsilon = inv^x_\varepsilon(u)$ given by relation



The next axiom states that T0, T1, T2 are sufficient for determining the class $\mathcal{T}(X)$ and the equivalence relation $\approx$.

Axiom T3: *The class $\mathcal{T}(X)$ is the smallest class of trees obtained by grafting of trees listed in axiom T0, and satisfsying axioms T1, T2. Moreover, two trees from $\mathcal{T}(X)$ are equivalent if and only if they can be proved equivalent after a finite string of applications of axioms T1, T2.*

4.2 First consequences

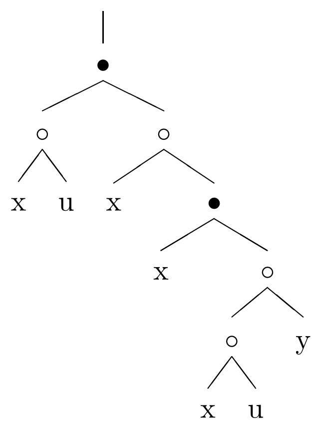
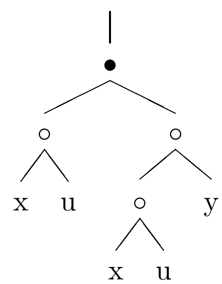
We shall use the axioms in order to obtain results that we shall use later, for dilatation structures.

Proposition 3. *For any x, u, y and v we have*

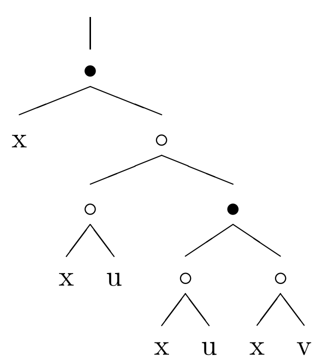
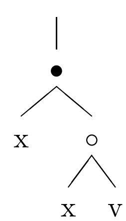
(a) $\Delta_\varepsilon^x(u, \Sigma_\varepsilon^x(u, y)) = y$,

(b) $\Sigma_\varepsilon^x(u, \Delta_\varepsilon^x(u, v)) = v$.

Proof. Indeed, for (a) we compute, using the definition 11 of the sum and difference trees, and axiom T1 several times.

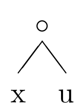
 $=$  $= y$

For (b) we proceed in the same way:

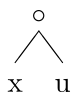
 $=$  $= v$.

The proof is done. □

Proposition 4. *We have the relations:*

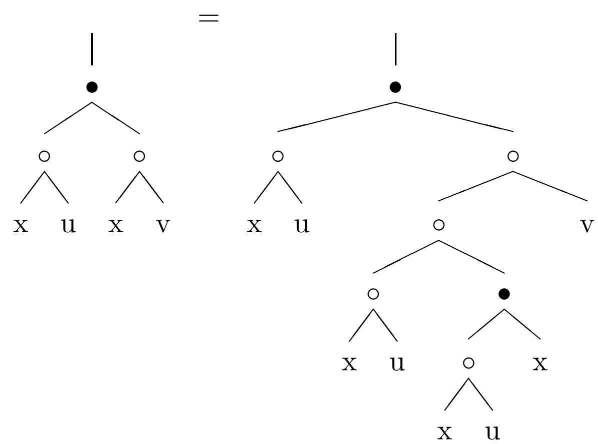
$\Delta_\varepsilon^x(u, v) = \Sigma_\varepsilon$  $(inv_\varepsilon^x(u), v)$, (4.3)

$$inv_\varepsilon^x(u) = \Delta_\varepsilon^x(u, x), \tag{4.4}$$



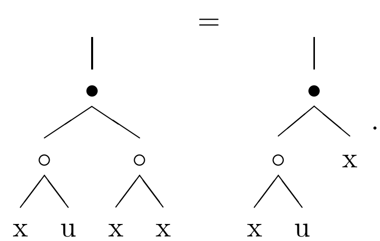
$$inv_\varepsilon \quad ((inv_\varepsilon^x(u)) = u. \tag{4.5}$$

Proof. Graphically, the relation (4.3) is:



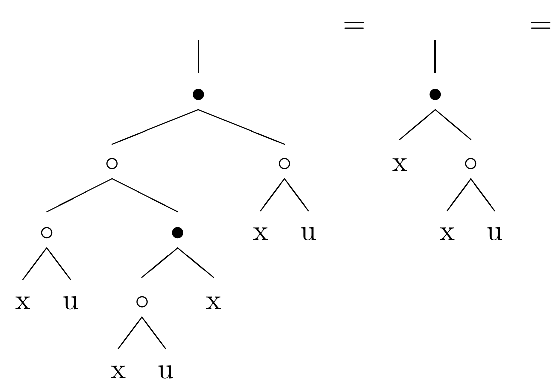
This is true by axiom T1.

The relation (4.4) is



This is true by axiom T2.

We prove the relation (4.5) by a string of equalities, starting from the left hand side to the right:



$= u.$

Here we have used axiom T1 several times. □

The relation (4.5) in last proposition shows that the "inverse function" inv_ε^x is not involutive, but shifted involutive.

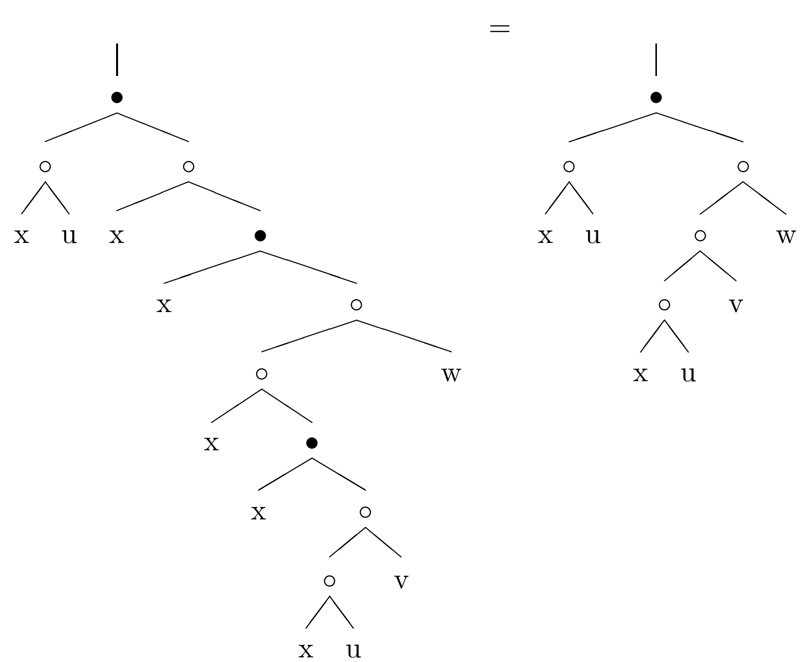
The next proposition proves that the function $\Sigma_\varepsilon^x(\cdot,\cdot)$ satisfies a shifted associativity property.

Proposition 5. *We have the relations:*

$$\Delta_{\varepsilon}^{x}(u, \Sigma_{\varepsilon}^{x}(\Sigma_{\varepsilon}^{x}(u,v),w)) = \Sigma_{\varepsilon}^{\begin{smallmatrix} & \circ & \\ \mathrm{x} & & \mathrm{u}\end{smallmatrix}}(v,w), \tag{4.6}$$

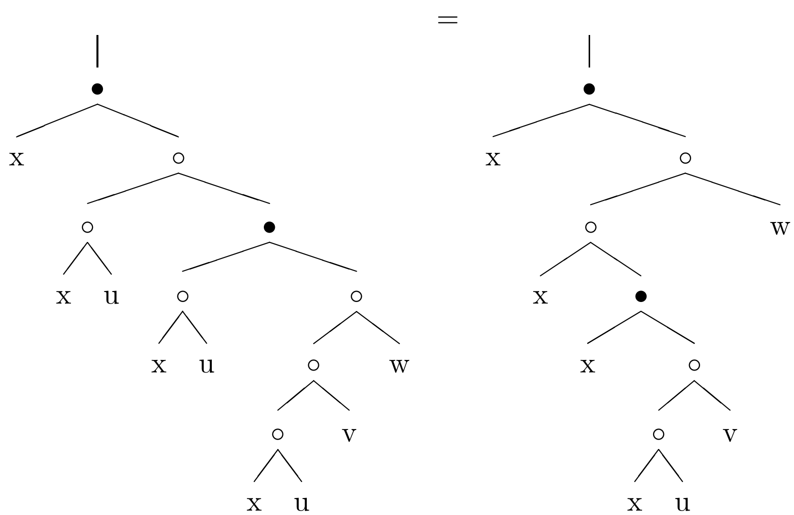
$$\Sigma_{\varepsilon}^{x}\left(u, \Sigma_{\varepsilon}^{\begin{smallmatrix} & \circ & \\ \mathrm{x} & & \mathrm{u}\end{smallmatrix}}(v,w)\right) = \Sigma_{\varepsilon}^{x}(\Sigma_{\varepsilon}^{x}(u,v),w). \tag{4.7}$$

Proof. Graphically, the relation (4.6) is:



This is true by axiom T1.

The relation (4.7) is is equivalent to (4.6), by proposition 3. We can also give a direct proof by graphically representing the relation:



This is true by axiom T1. □

5 Dilatation structures

The space (X, d) is a complete, locally compact metric space. This means that as a metric space (X, d) is complete and that small balls are compact.

5.1 Axioms of dilatation structures

The axioms of a dilatation structure (X, d, δ) are listed further. The first axiom is merely a preparation for the next axioms. That is why we counted it as axiom 0.

A0. Depending on the parameter $\varepsilon \in (0, +\infty)$, dilatations are objects having the following description.

For any $\varepsilon \in (0, 1]$ the dilatations are functions

$$\delta_\varepsilon^x : U(x) \to V_\varepsilon(x) \quad .$$

All such dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is $1 < A$ such that for any $x \in X$ we have

$$\bar{B}_d(x, A) \subset U(x) \ .$$

We suppose that for all $\varepsilon \in \Gamma$, $\varepsilon \in (0, 1)$, we have

$$B_d(x, \varepsilon) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset U(x) \ .$$

For $\varepsilon \in (1, +\infty)$ the associated dilatation

$$\delta_\varepsilon^x : W_\varepsilon(x) \to B_d(x, B) \ ,$$

is an injective, continuous, with continuous inverse on the image. We shall suppose that $W_\varepsilon(x)$ is open,

$$V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$$

and that for all $\varepsilon \in [0,1]$ and $u \in U(x)$ we have

$$\delta^x_{\varepsilon^{-1}} \ \delta^x_\varepsilon u \ = \ u \ .$$

We remark that we have the following string of inclusions, for any $\varepsilon \in (0,1]$ and any $x \in X$:

$$B_d(x,\varepsilon) \subset \delta^x_\varepsilon B_d(x,A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta^x_\varepsilon B_d(x,B) \quad .$$

A further technical condition on the sets $V_\varepsilon(x)$ and $W_\varepsilon(x)$ will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)

A1. We have $\delta^x_\varepsilon x = x$ for any point x. We also have $\delta^x_1 = id$ for any $x \in X$.

Let us define the topological space

$$dom\,\delta = \{(\varepsilon,x,y) \in (0,\infty) \times X \times X : \quad \text{if } \varepsilon \in (0,1] \text{ then } y \in U(x) \ , \text{ else } y \in W_\varepsilon(x)\} \quad ,$$

with the topology inherited from the product topology on $\Gamma \times X \times X$. Consider also $Cl(dom\,\delta)$, the closure of $dom\,\delta$ in $[0,\infty) \times X \times X$ with product topology. The function

$$\delta : dom\,\delta \to X$$

defined by $\delta(\varepsilon,x,y) = \delta^x_\varepsilon y$ is continuous. Moreover, it can be continuously extended to $Cl(dom\,\delta)$ and we have

$$\lim_{\varepsilon\to 0} \delta^x_\varepsilon y \ = \ x \quad .$$

A2. For any $x, \in K$, $\varepsilon, \mu \in \Gamma_1$ and $u \in \bar{B}_d(x,A)$ we have:

$$\delta^x_\varepsilon \delta^x_\mu u = \delta^x_{\varepsilon\mu} u \ .$$

A3. For any x there is a function $(u,v) \mapsto d^x(u,v)$, defined for any u,v in the closed ball (in distance d) $\bar{B}(x,A)$, such that

$$\lim_{\varepsilon\to 0} \quad \sup \left\{ \mid \frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \ - \ d^x(u,v) \mid \ : \ u,v \in \bar{B}_d(x,A) \right\} \ = \ 0$$

uniformly with respect to x in compact set.

Remark 12. The "distance" d^x can be degenerated. That means: there might be $v,w \in \bar{B}_d(x,A)$ such that $d^x(v,w) = 0$ but $v \neq w$. We shall use further the name "distance" for d^x, essentially by commodity, but keep in mind the possible degeneracy of d^x.

For the following axiom to make sense we impose a technical condition on the co-domains $V_\varepsilon(x)$: for any compact set $K \subset X$ there are $R = R(K) > 0$ and $\varepsilon_0 = \varepsilon(K) \in (0,1)$ such that for all $u,v \in \bar{B}_d(x,R)$ and all $\varepsilon \in \Gamma$, $\nu(\varepsilon) \in (0,\varepsilon_0)$, we have

$$\delta^x_\varepsilon v \in W_{\varepsilon^{-1}}(\delta^x_\varepsilon u) \ .$$

With this assumption the following notation makes sense:

$$\Delta^x_\varepsilon(u,v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v. \tag{5.1}$$

The next axiom can now be stated:

A4. We have the limit

$$\lim_{\varepsilon \to 0} \Delta_{\varepsilon}^{x}(u,v) = \Delta^{x}(u,v)$$

uniformly with respect to x, u, v in compact set.

Remark that with the tree notation we may identify (5.1) with the difference tree from definition 11 (a).

Definition 13. A triple (X, d, δ) which satisfies A0, A1, A2, A3, but d^x is degenerate for some $x \in X$, is called degenerate dilatation structure.

If the triple (X, d, δ) satisfies A0, A1, A2, A3 and d^x is non-degenerate for any $x \in X$, then we call it a weak dilatation structure.

If a weak dilatation structure satisfies A4 then we call it dilatation structure.

Note that it could be assumed, without great modification of the axioms, that:

(a) we may replace $(0, \infty)$ by Γ, a topological separated commutative group endowed with a continuous group morphism $\nu : \Gamma \to (0, +\infty)$ with $\inf \nu(\Gamma) = 0$. Here $(0, +\infty)$ is taken as a group with multiplication. The neutral element of Γ is denoted by 1. We use the multiplicative notation for the operation in Γ.

The morphism ν defines an invariant topological filter on Γ (equivalently, an end). Indeed, this is the filter generated by the open sets $\nu^{-1}(0, a)$, $a > 0$. From now on we shall name this topological filter (end) by "0" and we shall write $\varepsilon \in \Gamma \to 0$ for $\nu(\varepsilon) \in (0, +\infty) \to 0$.

The set $\Gamma_1 = \nu^{-1}(0, 1]$ is a semigroup. We note $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$ On the set $\bar{\Gamma} = \Gamma \cup \{0\}$ we extend the operation on Γ by adding the rules $00 = 0$ and $\varepsilon 0 = 0$ for any $\varepsilon \in \Gamma$. This is in agreement with the invariance of the end 0 with respect to translations in Γ.

In axioms A0, A1 we therefore may replace $[0, 1]$ by $\bar{\Gamma}_1$, and so forth.

(b) we may leave some flexibility in axiom A1 for the choice of base point of the dilatation, in the sense that

$$\lim_{\nu(\varepsilon) \to 0} \frac{1}{\nu(\varepsilon)} d(x, \delta_{\varepsilon}^{x} x) \ = \ 0$$

uniformly with respect to $x \in K$ compact set,

(c) we may relax the semigroup condition in the axiom A2, in the sense: for any compact set $K \subset X$, for any $x, \in K$, ε, μ with $\nu(\varepsilon), \nu(\mu) \in (0, 1)$ and $u, v \in \bar{B}_d(x, A)$ we have:

$$\frac{1}{\nu(\varepsilon\mu)} \mid d(\delta_{\varepsilon}^{x} \delta_{\mu}^{x} u, \delta_{\varepsilon}^{x} \delta_{\mu}^{x} v) \ - \ d(\delta_{\varepsilon\mu}^{x} u, \delta_{\varepsilon\mu}^{x} v) \mid \leq \ \mathcal{O}(\varepsilon\mu) \ .$$

(d) in axioms A3 and A4 we may replace "$\varepsilon \to 0$" by "$\nu(\varepsilon) \to 0$" and "$1/\varepsilon$" by "$1/\nu(\varepsilon)$".

We shall write the proofs of further results such that they work even if we modify the axioms in the sense explained above. We shall nevertheless use ε and not $\nu(\varepsilon)$, in order to avoid a too heavy notation.

The axioms, as given in this section, are said to be in strong form. With the modifications explained at points (a),(b), (c), (d) above, the axioms are said to be in weak form.

Further, axioms are taken in weak form, with the notational conventions explained above, unless it is explicitely stated that some axiom has to be taken in strong form.

5.2 Dilatation structures, tangent cones and metric profiles

We shall explain now what the axioms mean. The first axiom A1 is stating that the distance between $\delta^x_\varepsilon x$ and x is negligible with respect to ε. If $\delta^x_\varepsilon x = x$ then this axiom is trivially satisfied.

The second axiom A2. states that in an approximate sense the transformations δ^x_ε form an action of Γ on X. As previously, if we suppose that

$$\delta^x_\varepsilon \delta^x_\mu \ = \ \delta^x_{\varepsilon\mu}$$

then this axiom is trivially satisfied.

Remark now that the binary tree formalism described in section 4 underlies and simplifies the calculus with dilatation structures. More precisely, we shall use the results in section 4 in the proof of theorems in the next section.

The notation with binary trees for composition of dilatations is not directly adapted for taking limits as $\varepsilon \to 0$. An extension of the formalism can be made in this direction, but this would add length to this paper, which is devoted to first properties of dilatation structures. We reserve the full description of the formalism for a future paper.

In axiom A3 we take limits. In this subsection we shall look at dilatation structures from the metric point of view, by using Gromov-Hausdorff distance and metric profiles.

We state the interpretation of the axiom A3 as a theorem. But before a definition: we denote by (δ, ε) the distance on

$$\bar{B}_{d^x}(x,1) \ = \ \{ y \in X \, : \, d^x(x,y) \leq 1 \}$$

given by

$$(\delta, \varepsilon)(u,v) \ = \ \frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) \ .$$

Theorem 6. *Let (X,d,δ) be a dilatation structure. The following are consequences of axioms A0, ... , A3 only:*

(a) for all $u,v \in X$ such that $d(x,u) \leq 1$ and $d(x,v) \leq 1$ and all $\mu \in (0,A)$ we have:

$$d^x(u,v) \ = \ \frac{1}{\mu} d^x(\delta^x_\mu u, \delta^x_\mu v) \ .$$

We shall say that d^x has the cone property with respect to dilatations.

(b) The curve $\varepsilon > 0 \mapsto \mathbb{P}^x(\varepsilon) \ = \ [\bar{B}_{d^x}(x,1), (\delta,\varepsilon), x]$ is a metric profile.

Proof. (a) Indeed, for $\varepsilon, \mu \in (0,1)$ we have:

$$\mid \frac{1}{\varepsilon\mu} d(\delta^x_\varepsilon \delta^x_\mu u, \delta^x_\varepsilon \delta^x_\mu v) \ - \ d^x(u,v) \mid \leq \mid \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} u, \delta^x_\varepsilon \delta^x_\mu u) - \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} v, \delta^x_\varepsilon \delta^x_\mu v) \mid +$$

$$+ \ \mid \frac{1}{\varepsilon\mu} d(\delta^x_{\varepsilon\mu} u, \delta^x_{\varepsilon\mu} v) \ - \ d^x(u,v) \mid \ .$$

Use now the axioms A2 and A3 and pass to the limit with $\varepsilon \to 0$. This gives the desired equality.

(b)We have to prove that $\mathbb{P}^x$ is a metric profile. For this we have to compare two pointed metric spaces:

$$\big((\delta^x, \varepsilon\mu), \bar{B}_{d^x}(x,1), x \big) \quad \text{and} \quad \left(\frac{1}{\mu}(\delta^x, \varepsilon), \bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,1), x \right) \ .$$

Let $u \in X$ such that

$$\frac{1}{\mu}(\delta^x, \varepsilon)(x,u) \leq 1 \ .$$

This means that:

$$\frac{1}{\varepsilon} d(\delta_\varepsilon^x x, \delta_\varepsilon^x u) \ \leq \ \mu \ .$$

Use further axioms A1, A2 and the cone property proved before:

$$\frac{1}{\varepsilon} d^x(\delta_\varepsilon^x x, \delta_\varepsilon^x u) \ \leq \ (\mathcal{O}(\varepsilon) + 1)\mu$$

therefore

$$d^x(x,u) \ \leq \ (\mathcal{O}(\varepsilon) + 1)\mu \ .$$

It follows that for any $u \in \bar{B}_{\frac{1}{\mu}(\delta^x, \varepsilon)}(x,1)$ we can choose $w(u) \in \bar{B}_{d^x}(x,1)$ such that

$$\frac{1}{\mu} d^x(u, \delta_\mu^x w(u)) \ = \ \mathcal{O}(\varepsilon) \ .$$

We want to prove that

$$\mid \frac{1}{\mu}(\delta^x, \varepsilon)(u_1, u_2) \ - \ (\delta^x, \varepsilon\mu)(w(u_1), w(u_2)) \mid \leq \ \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon) \ .$$

This goes as following:

$$\mid \frac{1}{\mu}(\delta^x, \varepsilon)(u_1, u_2) \ - \ (\delta^x, \varepsilon\mu)(w(u_1), w(u_2)) \mid$$

$$= \mid \frac{1}{\varepsilon\mu} d(\delta_\varepsilon^x u_1, \delta_\varepsilon^x u_2) \ - \ \frac{1}{\varepsilon\mu} d(\delta_{\varepsilon\mu}^x w(u_1), \delta_{\varepsilon\mu}^x w(u_2)) \mid \leq$$

$$\leq \ \mathcal{O}(\varepsilon\mu) \ + \ \mid \frac{1}{\varepsilon\mu} d(\delta_\varepsilon^x u_1, \delta_\varepsilon^x u_2) \ - \ \frac{1}{\varepsilon\mu} d(\delta_\varepsilon^x \delta_\mu^x w(u_1), \delta_\varepsilon^x \delta_\mu^x w(u_2)) \mid \leq$$

$$\leq \ \mathcal{O}(\varepsilon\mu) \ + \ \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} \mid d^x(u_1, u_2) \ - \ d^x(\delta_\mu^x w(u_1), \delta_\mu^x w(u_2)) \ .$$

In order to obtain the last estimate we used twice axiom A3. We continue:

$$\mathcal{O}(\varepsilon\mu) \ + \ \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} \mid d^x(u_1, u_2) \ - \ d^x(\delta_\mu^x w(u_1), \delta_\mu^x w(u_2)) \ \leq$$

$$\leq \mathcal{O}(\varepsilon\mu) \ + \ \frac{1}{\mu}\mathcal{O}(\varepsilon) \ + \ \frac{1}{\mu} d^x(u_1, \delta_\mu^x w(u_1)) \ + \ \frac{1}{\mu} d^x(u_1, \delta_\mu^x w(u_2)) \ \leq$$

$$\leq \ \mathcal{O}(\varepsilon\mu) + \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon) \ .$$

This shows that the property (b) of a metric profile is satisfied. The property (a) is proved in theorem 7. □

The following theorem is related to Mitchell [12] theorem 1, concerning sub-riemannian geometry.

Theorem 7. *In the hypothesis of theorem 6, we have the following limit:*

$$\lim_{\varepsilon \to 0} \ \frac{1}{\varepsilon} \sup \left\{ \mid d(u,v) - d^x(u,v) \mid \ : d(x,u) \leq \varepsilon \ , \ d(x,v) \leq \varepsilon \right\} \ = \ 0 \ .$$

Therefore if d^x is a true (i.e. nondegenerate) distance, then (X,d) admits a metric tangent space in x.

Moreover, the metric profile $[\bar{B}_{d^x}(x,1),(\delta,\varepsilon),x]$ *is almost nice, in the following sense: let* $c\in(0,1)$*. Then we have the inclusion:*

$$\delta^x_{\mu^{-1}}\left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)\right) \subset \bar{B}_{d^x}(x,1) \quad .$$

Moreover, the following Gromov-Hausdorff distance is of order $\mathcal{O}(\varepsilon)$ *for* μ *fixed (that is the modulus of convergence* $\mathcal{O}(\varepsilon)$ *does not depend on* μ*) :*

$$\mu\ d_{GH}\left([\bar{B}_{d^x}(x,1),(\delta^x,\varepsilon),x],\ [\delta^x_{\mu^{-1}}\left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)\right),(\delta^x,\varepsilon\mu),x]\right)=\mathcal{O}(\varepsilon)\ .$$

For another Gromov-Hausdorff distance we have the estimate:

$$d_{GH}\left([\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c),\frac{1}{\mu}(\delta^x,\varepsilon),x]\ ,\ [\delta^x_{\mu^{-1}}\left(\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)\right),(\delta^x,\varepsilon\mu),x]\right)=\mathcal{O}(\varepsilon\mu)$$

when $\varepsilon\in(0,\varepsilon(c))$*.*

Proof. We start from the axioms A0, A3 and we use the cone property. By A0, for $\varepsilon\in(0,1)$ and $u,v\in\bar{B}_d(x,\varepsilon)$ there exist $U,V\in\bar{B}_d(x,A)$ such that

$$u=\delta^x_\varepsilon U, v=\delta^x_\varepsilon V.$$

By the cone property we have

$$\frac{1}{\varepsilon}\mid d(u,v)-d^x(u,v)\mid=\mid\frac{1}{\varepsilon}d(\delta^x_\varepsilon U,\delta^x_\varepsilon V)-d^x(U,V)\mid .$$

By A2 we have

$$\mid\frac{1}{\varepsilon}d(\delta^x_\varepsilon U,\delta^x_\varepsilon V)-d^x(U,V)\mid\leq\mathcal{O}(\varepsilon).$$

This proves the first part of the theorem.

For the second part of the theorem take any $u\in\bar{B}_{\frac{1}{\mu}(\delta^x,\varepsilon)}(x,c)$. We have then

$$d^x(x,u)\ \leq\ c\mu+\mathcal{O}(\varepsilon)\ .$$

Then there exists $\varepsilon(c)>0$ such that for any $\varepsilon\in(0,\varepsilon(c))$ and u in the mentioned ball we have:

$$d^x(x,u)\leq\ \mu$$

In this case we can take directly $w(u)=\delta^x_{\mu^{-1}}u$ and simplify the string of inequalities from the proof of theorem 6, point (b), to get eventually the three points from the second part of the theorem. □

6 Tangent bundle of a dilatation structure

In this section we shall use the calculus with binary decorated trees introduced in section 4, for a space endowed with a dilatation structure.

6.1 Main results

Theorem 8. *Let (X, d, δ) be a dilatation structure. Then the "infinitesimal translations"*

$$L_u^x(v) = \lim_{\varepsilon \to 0} \Delta_\varepsilon^x(u, v)$$

are d^x isometries.

Proof. The first part of the conclusion of theorem 7 can be written as:

$$\sup \left\{ \frac{1}{\varepsilon} \mid d(u,v) \ - \ d^x(u,v) \mid \ : \ d(x,u) \leq \frac{3}{2}\varepsilon, \ d(x,v) \leq \frac{3}{2}\varepsilon \right\} \to 0 \tag{6.1}$$

as $\varepsilon \to 0$.

For $\varepsilon > 0$ sufficiently small the points $x, \delta_\varepsilon^x u, \delta_\varepsilon^x v, \delta_\varepsilon^x w$ are close one to another. Precisely, we have

$$d(\delta_x^\varepsilon u, \delta_x^\varepsilon v) = \varepsilon (d^x(u,v) + \mathcal{O}(\varepsilon)) \quad .$$

Therefore, if we choose u, v, w such that $d^x(u,v) < 1, d^x(u,w) < 1$, then there is $\eta > 0$ such that for all $\varepsilon \in (0, \eta)$ we have

$$d(\delta_x^\varepsilon u, \delta_x^\varepsilon v) \leq \frac{3}{2}\varepsilon \quad , \quad d(\delta_x^\varepsilon u, \delta_x^\varepsilon v) \leq \frac{3}{2}\varepsilon \quad .$$

We apply the estimate (6.1) for the basepoint $\delta_\varepsilon^x u$ to get:

$$\frac{1}{\varepsilon} \mid d(\delta_\varepsilon^x v, \delta_\varepsilon^x w) \ - \ d^{\delta_\varepsilon^x u}(\delta_\varepsilon^x v, \delta_\varepsilon^x w) \mid \to 0$$

when $\varepsilon \to 0$. This can be written, using the cone property of the distance $d^{\delta_\varepsilon^x u}$, like this:

$$\mid \frac{1}{\varepsilon} d(\delta_\varepsilon^x v, \delta_\varepsilon^x w) \ - \ d^{\delta_\varepsilon^x u}\left(\delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v, \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x w \right) \mid \to 0 \tag{6.2}$$

as $\varepsilon \to 0$. By the axioms A1, A3, the function

$$(x, u, v) \mapsto d^x(u, v)$$

is an uniform limit of continuous functions, therefore uniformly continuous on compact sets. We can pass to the limit in the left hand side of the estimate (6.2), using this uniform continuity and axioms A3, A4, to get the result. □

Let us define, in agreement with definition 11 (b):

$$\Sigma_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^x \delta_\varepsilon^{\delta_\varepsilon^x u} v.$$

Corollary 9. *If for any x the distance d^x is non degenerate then there exists $C > 0$ such that: for any x and u with $d(x,u) \leq C$ there exists a d^x isometry $\Sigma^x(u, \cdot)$ obtained as the limit:*

$$\lim_{\varepsilon \to 0} \Sigma_\varepsilon^x(u, v) = \Sigma^x(u, v)$$

uniformly with respect to x, u, v in compact set.

Proof. From theorem 8 we know that $\Delta^x(u, \cdot)$ is a d^x isometry. If d^x is non degenerate then $\Delta^x(u, \cdot)$ is invertible. Let $\Sigma^x(u, \cdot)$ be the inverse.

From proposition 3 we know that $\Sigma_\varepsilon^x(u, \cdot)$ is the inverse of $\Delta_\varepsilon^x(u, \cdot)$. Therefore

$$d^x(\Sigma_\varepsilon^x(u, w), \Sigma^x(u, w)) = d^x(\Delta^x(u, \Sigma_\varepsilon^x(u, w)), w) =$$

$$= d^x(\Delta^x(u, \Sigma_\varepsilon^x(u, w)), \Delta_\varepsilon^x(u, \Sigma_\varepsilon^x(u, w)).$$

From the uniformity of convergence in theorem 8 and the uniformity assumptions in axioms of dilatation structures, the conclusion follows. □

The next theorem is the generalization of proposition 2. It is the main result of this paper.

Theorem 10. *Let (X, d, δ) be a dilatation structure (which satisfies the strong form of the axiom A2), such that for any $x \in X$ the distance d^x is non degenerate. Then for any $x \in X$ $(U(x), \Sigma^x, \delta^x)$ is a conical group. Moreover, left translations of this group are d^x isometries.*

Proof. We start by proving that $(U(x), \Sigma^x)$ is a local uniform group. The uniformities are induced by the distance d.

We shall use the general relations written in terms of binary decorated trees. Indeed, according to relation (4.4) in proposition 4, we can pass to the limit with $\varepsilon \to 0$ and define:

$$inv^x(u) = \lim_{\varepsilon \to 0} \Delta^x_\varepsilon(u, x) = \Delta^x(u, x).$$

From relation (4.5) we get (after passing to the limit with $\varepsilon \to 0$)

$$inv^x(inv^x(u)) = u.$$

We shall see that $inv^x(u)$ is the inverse of u. Relation (4.3) gives:

$$\Delta^x(u, v) = \Sigma^x(inv^x(u), v) \tag{6.3}$$

therefore relations (a), (b) from proposition 3 give

$$\Sigma^x(inv^x(u), \Sigma^x(u, v)) = v, \tag{6.4}$$

$$\Sigma^x(u, \Sigma^x(u, v)) = v. \tag{6.5}$$

Relation (4.7) from proposition 5 gives

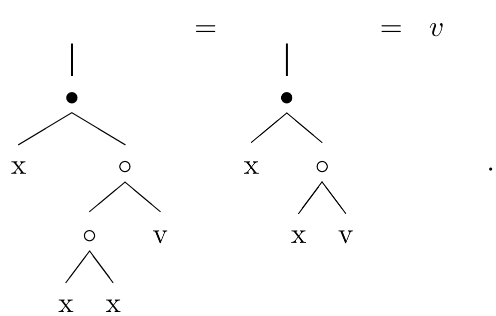
$$\Sigma^x(u, \Sigma^x(v, w)) = \Sigma^x(\Sigma^x(u, v), w) \tag{6.6}$$

which shows that Σ^x is an associative operation. From (6.5), (6.4) we obtain that for any u, v

$$\Sigma^x(\Sigma^x(inv^x(u), u), v) = v, \tag{6.7}$$

$$\Sigma^x(\Sigma^x(u, inv^x(u)), v) = v. \tag{6.8}$$

Remark that for any x, v and $\varepsilon \in (0, 1)$ we have $\Sigma^x(x, v) = v$. indeed, this means:



Therefore x is a neutral element at left for the operation Σ^x. From the definition of inv^x, relation (6.3) and the fact that inv^x is equal to its inverse, we get that x is an inverse at right too: for any x, v we have

$$\Sigma^x(v, x) = v.$$

Replace now v by x in relations (6.7), (6.8) and prove that indeed $inv^x(u)$ is the inverse of u.

We still have to prove that $(U(x), \Sigma^x)$ admits δ^x as dilatations.In this reasoning we need the axiom A2 in strong form.

Namely we have to prove that for any $\mu \in (0,1)$ we have

$$\delta_\mu^x \Sigma^x(u,v) = \Sigma^x(\delta_\mu^x u, \delta_\mu^x v).$$

For this is sufficient to notice that

$$\Delta_\varepsilon^x \left(\delta_\mu^x u, \delta_\mu^x v\right) = \delta_\mu^{\delta_{\epsilon\mu}^x u} \Delta_{\varepsilon\mu}^x(u,v) \quad .$$

and pass to the limit as $\varepsilon \to 0$. Notice that here we used the fact that dilatations δ_ε^x and δ_μ^x exactly commute (axiom A2 in strong form).

Finally, left translations L_u^x are d^x isometries. Indeed, this is a straightforward consequence of theorem 8 and corollary 9. □

The conical group $(U(x), \Sigma^x, \delta^x)$ can be regarded as the tangent space of (X, δ, d) at x and denoted further by $T_x X$.

6.2 Algebraic interpretation

In order to better understand the algebraic structure of the sum, difference, inverse operations induced by a dilatation structure, we collect previous results regarding the properties of these operations, into one place.

Theorem 11. *Let (X, d, δ) be a weak dilatation structure. Then, for any $x \in X$, $\varepsilon \in \Gamma$, $\nu(\varepsilon) < 1$, we have:*

(a) for any $u \in U(x)$, $\Sigma_\varepsilon^x(x,u) = u$.

(b) for any $u \in U(x)$ the functions $\Sigma_\varepsilon^x(u,\cdot)$ and $\Delta_\varepsilon^x(u,\cdot)$ are inverse one to another.

(c) the inverse function is shifted involutive: for any $u \in U(x)$,

$$inv_\varepsilon^{\delta_\varepsilon^x u}\, inv_\varepsilon^x(u) = u \quad .$$

(d) the sum operation is shifted associative: for any u, v, w sufficiently close to x we have

$$\Sigma_\varepsilon^x \left(u, \Sigma_\varepsilon^{\delta_\varepsilon^x u}(v,w)\right) = \Sigma_\varepsilon^x(\Sigma^x(u,v), w) \quad .$$

(e) the difference, inverse and sum operations are related by

$$\Delta_\varepsilon^x(u,v) = \Sigma_\varepsilon^{\delta_\varepsilon^x u}\left(inv_\varepsilon^x(u), v\right) \quad ,$$

for any u, v sufficiently close to x.

(f) for any u, v sufficiently close to x and $\mu \in \Gamma$, $\nu(\mu) < 1$, we have:

$$\Delta_\varepsilon^x \left(\delta_\mu^x u, \delta_\mu^x v\right) = \delta_\mu^{\delta_{\varepsilon\mu}^x u} \Delta_{\varepsilon\mu}^x(u,v) \quad .$$

7 Dilatation structures and differentiability

7.1 Equivalent dilatation structures

Definition 14. Two dilatation structures (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ are equivalent if

(a) the identity map $id : (X, d) \to (X, \overline{d})$ is bilipschitz and

(b) for any $x \in X$ there are functions P^x, Q^x (defined for $u \in X$ sufficiently close to x) such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \overline{d}\left(\delta_\varepsilon^x u, \overline{\delta}_\varepsilon^x Q^x(u)\right) = 0, \tag{7.1}$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d\left(\overline{\delta}_\varepsilon^x u, \delta_\varepsilon^x P^x(u)\right) = 0, \tag{7.2}$$

uniformly with respect to x, u in compact sets.

Proposition 12. *Two dilatation structures (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ are equivalent if and only if*

(a) the identity map $id : (X, d) \to (X, \overline{d})$ is bilipschitz and

(b) for any $x \in X$ there are functions P^x, Q^x (defined for $u \in X$ sufficiently close to x) such that

$$\lim_{\varepsilon \to 0} \left(\overline{\delta}_\varepsilon^x\right)^{-1} \delta_\varepsilon^x(u) = Q^x(u), \tag{7.3}$$

$$\lim_{\varepsilon \to 0} \left(\delta_\varepsilon^x\right)^{-1} \overline{\delta}_\varepsilon^x(u) = P^x(u), \tag{7.4}$$

uniformly with respect to x, u in compact sets.

Proof. We make the notations

$$Q_\varepsilon^x(u) = \left(\overline{\delta}_\varepsilon^x\right)^{-1} \delta_\varepsilon^x(u),$$

$$P_\varepsilon^x(u) = \left(\delta_\varepsilon^x\right)^{-1} \overline{\delta}_\varepsilon^x(u).$$

The relation (7.1) is equivalent to

$$\mathcal{O}(\varepsilon) + \overline{d}^x\left(Q_\varepsilon^x(u), Q^x(u)\right) \to 0,$$

$$\mathcal{O}(\varepsilon) + d^x\left(P_\varepsilon^x(u), P^x(u)\right) \to 0$$

as $\varepsilon \to 0$, uniformly with respect to x, u in compact sets. The conclusion follows after passing $\varepsilon \to 0$. □

The next theorem shows a link between the tangent bundles of equivalent dilatation structures.

Theorem 13. *Let (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ be equivalent dilatation structures. Suppose that for any $x \in X$ the distance d^x is non degenerate. Then for any $x \in X$ and any $u, v \in X$ sufficiently close to x we have:*

$$\overline{\Sigma}^x(u, v) = Q^x\left(\Sigma^x\left(P^x(u), P^x(v)\right)\right). \tag{7.5}$$

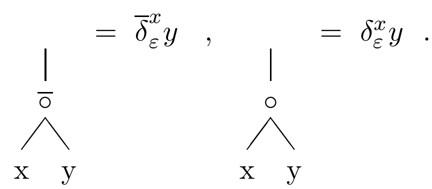
The two tangent bundles are therefore isomorphic in a natural sense.

Proof. We notice first that the hypothesis is symmetric: if d^x is non degenerate then $\overline{d}^x$ is non degenerate too. Indeed, this is straightforward from definition 14 (a) and axiom A3 for the two dilatation structures.

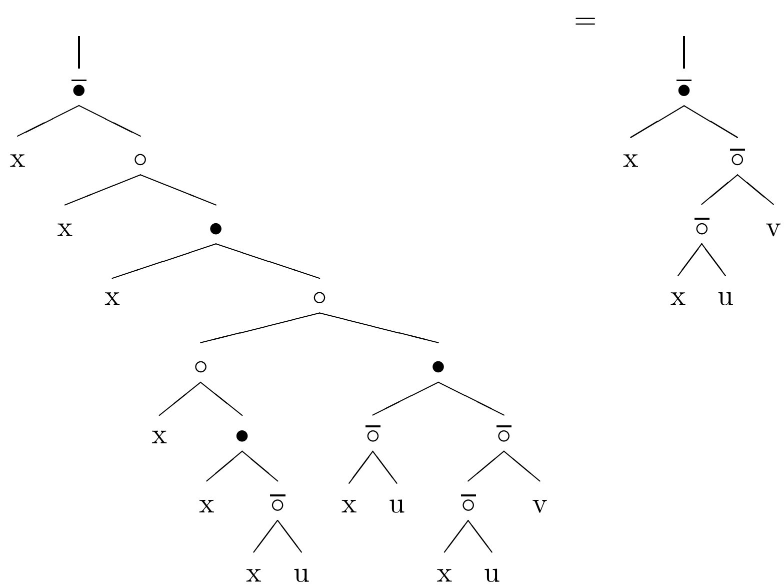
For the proof of relation (7.5) is enough to remark that for $\varepsilon > 0$ but sufficiently small we have

$$\overline{\Sigma}_\varepsilon^x(u,v) = Q_\varepsilon^x\left(\Sigma_\varepsilon^x\left(P_\varepsilon^x(v), P_\varepsilon^{\overline{\delta}_\varepsilon^x u}(v)\right)\right) . \tag{7.6}$$

Indeed, with tree notation, let



The relation (7.6), written from right to left, is:



But this is true by cancellations of dilatations and definitions of the operators P_ε^x and Q_ε^x. $\square$

7.2 Differentiable functions

Dilatation structures allow to define differentiable functions. The idea is to keep only one relation from definition 14, namely (7.1). We also renounce to uniform convergence with respect to x and u, and we replace this with uniform convergence in$u$ and with a conical group morphism condition for the derivative.

First we need the natural definition below.

Definition 15. Let (N, δ) and $(M, \bar{\delta})$ be two conical groups. A function $f : N \to M$ is a conical group morphism if f is a group morphism and for any $\varepsilon > 0$ and $u \in N$ we have $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$.

The definition of derivative with respect to dilatations structures follows.

Definition 16. Let (X, δ, d) and $(Y, \overline{\delta}, \overline{d})$ be two dilatation structures and $f : X \to Y$ be a continuous function. The function f is differentiable in x if there exists a conical group morphism $Q^x : T_x X \to T_{f(x)} Y$, defined on a neighbourhood of x with values in a neighbourhood of $f(x)$ such that

$$\lim_{\varepsilon \to 0} \sup \left\{ \frac{1}{\varepsilon} \overline{d} \left(f\left(\delta_\varepsilon^x u\right), \overline{\delta}_\varepsilon^{f(x)} Q^x(u) \right) \ : \ d(x,u) \leq \varepsilon \right\} = 0, \tag{7.7}$$

The morphism Q^x is called the derivative of f at x and will be sometimes denoted by $Df(x)$.

The function f is uniformly differentiable if it is differentiable everywhere and the limit in (7.7) is uniform in x in compact sets.

This definition deserves a short discussion. Let (X, δ, d) and $(Y, \overline{\delta}, \overline{d})$ be two dilatation structures and $f : X \to Y$ a function differentiable in x. The derivative of f in x is a conical group morphism $Df(x) : T_x X \to T_{f(x)} Y$, which means that $Df(x)$ is defined on a open set around x with values in a open set around $f(x)$, having the properties:

(a) for any u, v sufficiently close to x

$$Df(x)\left(\Sigma^x(u,v)\right) = \Sigma^{f(x)}\left(Df(x)(u), Df(x)(v)\right),$$

(b) for any u sufficiently close to x and any $\varepsilon \in (0,1]$

$$Df(x)\left(\delta_\varepsilon^x u\right) = \overline{\delta}_\varepsilon^{f(x)}\left(Df(x)(u)\right),$$

(c) the function $Df(x)$ is continuous, as uniform limit of continuous functions. Indeed, the relation (7.7) is equivalent to the existence of the uniform limit (with respect to u in compact sets)

$$Df(x)(u) = \lim \varepsilon \to 0 \overline{\delta}_{\varepsilon^{-1}}^{f(x)}\left(f\left(\delta_\varepsilon^x u\right)\right).$$

From (7.7) alone and axioms of dilatation structures we can prove properties (b) and (c). We can reformulate therefore the definition of the derivative by asking that $Df(x)$ exists as an uniform limit (as in point (c) above) and that $Df(x)$ has the property (a) above. From these considerations the chain rule for derivatives is straightforward.

A trivial way to obtain a differentiable function (everywhere) is to modify the dilatation structure on the target space.

Definition 17. Let (X, δ, d) be a dilatation structure and $f : (X,d) \to (Y, \overline{d})$ be a bilipschitz and surjective function. We define then the transport of (X, δ, d) by f, named $(Y, f * \delta, \overline{d})$, by:

$$(f * \delta)_\varepsilon^{f(x)} f(u) = f\left(\delta_\varepsilon^x u\right).$$

The relation of differentiability with equivalent dilatation structures is given by the following simple proposition.

Proposition 14. *Let (X, δ, d) and $(X, \overline{\delta}, \overline{d})$ be two dilatation structures and $f : (X,d) \to (X, \overline{d})$ be a bilipschitz and surjective function. The dilatation structures $(X, \overline{\delta}, \overline{d})$ and $(X, f * \delta, \overline{d})$ are equivalent if and only if f and f^{-1} are uniformly differentiable.*

Proof. Straightforward from definitions 14 and 17. □

8 Differential structure, conical groups and dilatation structures

In this section we collect some facts which relate differential structures with dilatation structures. We resume then the paper with a justification of the unusual way of defining uniform groups (definition 7) by the fact that the *op* function (the group operation) is differentiable with respect to dilatation structures which are natural for a group with dilatations.

8.1 Differential structures and dilatation structures

A differential structure on a manifold is an equivalence class of compatible atlases. We show here that an atlas induces an equivalence class of dilatation structures and that two compatible atlases induce the same equivalence class of dilatation structures.

Let M be a $\mathcal{C}^1$ n-dimensional real manifold and $\mathcal{A}$ an atlas of this manifold. For each chart $\phi : W \subset M \to \mathbb{R}^n$ we shall define a dilatation structure on W.

Indeed, suppose that $\phi(W) \subset \mathbb{R}^n$ is convex (if not then take an open subset of W with this property). For $x, u \in W$ and $\varepsilon \in (0,1]$ define the dilatation

$$\delta_\varepsilon^x u = \phi^{-1}\left(\phi(x) + \varepsilon(\phi(u) - \phi(x))\right).$$

Otherwise said, the dilatations in W are transported from $\mathbb{R}^n$. Equally, we transport on W the euclidean distance of $\mathbb{R}^n$. We obviously get a dilatation structure on W.

If we have two charts $\phi_i : W_i \subset M \to \mathbb{R}^n$, $i = 1,2$, belonging to the same atlas $\mathcal{A}$, then we have two equivalent dilatation structures on $W_1 \cap W_2$. Indeed, the atlas $\mathcal{A}$ is $\mathcal{C}^1$ therefore the distances (induced from the charts) are (locally) in bilipschitz equivalence. Denote by $\overline{\delta}$ the dilatation obtained from the chart ϕ_2. A short computation shows that (we use here the transition map $\phi_{21} = \phi_2(\phi_1)^{-1}$):

$$Q_\varepsilon^x(u) = (\phi_2)^{-1}\left(\phi_2(x) + \frac{1}{\varepsilon}\left(\phi_{21}\left(\phi_1(x) + \varepsilon(f(u) - f(x))\right) - \phi_2(x)\right)\right),$$

therefore, as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} Q_\varepsilon^x(u) = Q^x(u) = (\phi_2)^{-1}\left(\phi_2(x) + D\phi_{21}(f(x))(f(u) - f(x))\right).$$

A similar computation shows that P^x also exists. The uniform convergence requirements come from the fact that we use a $\mathcal{C}^1$ atlas.

A similar reasoning shows that in fact two compatible atlases induce the same equivalence class of dilatation structures.

8.2 Conical groups and dilatation structures

In a group with dilatations (G, δ) we define dilatations based in any point $x \in G$ by

$$\delta_\varepsilon^x u = x\delta_\varepsilon(x^{-1}u). \tag{8.1}$$

Definition 18. A normed group with dilatations $(G, \delta, \|\cdot\|)$ is a group with dilatations (G, δ) endowed with a continuous norm function $\|\cdot\| : G \to \mathbb{R}$ which satisfies (locally, in a neighbourhood of the neutral element e) the properties:

(a) for any x we have $\|x\| \geq 0$; if $\|x\| = 0$ then $x = e$,

(b) for any x, y we have $\|xy\| \leq \|x\| + \|y\|$,

(c) for any x we have $\|x^{-1}\| = \|x\|$,

(d) the limit $\lim_{\varepsilon \to 0} \frac{1}{\nu(\varepsilon)} \|\delta_\varepsilon x\| = \|x\|^N$ exists, is uniform with respect to x in compact set,

(e) if $\|x\|^N = 0$ then $x = e$.

It is easy to see that if $(G, \delta, \| \cdot \|)$ is a normed group with dilatations then $(G, \beta, \delta, \| \cdot \|^N)$ is a normed conical group. The norm $\| \cdot \|^N$ satisfies the stronger form of property (d) definition 18: for any $\varepsilon > 0$

$$\|\delta_\varepsilon x\|^N = \varepsilon \|x\|^N .$$

Normed groups with dilatations can be encountered in sub-Riemannian geometry. Normed conical groups generalize the notion of Carnot groups.

In a normed group with dilatations we have a natural left invariant distance given by

$$d(x, y) = \|x^{-1} y\|. \tag{8.2}$$

Theorem 15. *Let $(G, \delta, \| \cdot \|)$ be a locally compact normed group with dilatations. Then (G, δ, d) is a dilatation structure, where δ are the dilatations defined by (8.1) and the distance d is induced by the norm as in (8.2).*

Proof. The axiom A0 is straightforward from definition 7, definition 8, axiom H0, and because the dilatation structure is left invariant, in the sense that the transport by left translations in G, according to definition 17, preserves the dilatations δ. We also trivially have axioms A1 and A2 satisfied.

For the axiom A3 remark that

$$d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) = d(x \delta_\varepsilon (x^{-1} u), x \delta_\varepsilon (x^{-1} u)) = d(\delta_\varepsilon (x^{-1} u), \delta_\varepsilon (x^{-1} v)).$$

Denote $U = x^{-1} u$, $V = x^{-1} v$ and for $\varepsilon > 0$ let

$$\beta_\varepsilon (u, v) = \delta_\varepsilon^{-1} \left((\delta_\varepsilon u)(\delta_\varepsilon v) \right) .$$

We have then:

$$\frac{1}{\varepsilon} d(\delta^x_\varepsilon u, \delta^x_\varepsilon v) = \frac{1}{\varepsilon} \|\delta_\varepsilon \beta_\varepsilon \left(\delta_\varepsilon^{-1} \left((\delta_\varepsilon V)^{-1} \right), U \right) \| \quad .$$

Define the function

$$d^x (u, v) = \|\beta(V^{-1}, U)\|^N .$$

From definition 8 axioms H1, H2, and from definition 18 (d), we obtain that axiom A3 is satisfied.

For the axiom A4 we have to compute:

$$\Delta^x (u, v) = \delta^{\delta^x_\varepsilon u}_{\varepsilon^{-1}} \delta^x_\varepsilon v = (\delta^x_\varepsilon u) \, (\delta_\varepsilon)^{-1} \left((\delta^x_\varepsilon u)^{-1} \, (\delta^x_\varepsilon v) \right) =$$

$$= (x \delta_\varepsilon U) \, \beta_\varepsilon \left(\delta_\varepsilon^{-1} \left((\delta_\varepsilon V)^{-1} \right), U \right) \to x \beta \left(V^{-1}, U \right)$$

as $\varepsilon \to 0$. Therefore axiom A4 is satisfied. □

We remarked in the proof of the previous theorem that the transport by left translations in G, according to definition 17, preserves the dilatation structure on G. This implies, according to proposition 14, that left translations are differentiable. On the contrary, a short computation and examples from sub-Riemannian geometry indicate that right translations are not differentiable.

Nevertheless, the operation *op* is differentiable, if we endow the group $G^{(2)} = G \times G$ with a good dilatation structure. This will justify the non standard way to define local uniform groups in definition 7.

Start from the fact that if G is a local uniform group then $G^{(2)}$ is a local uniform group too. If G is also normed, with dilatations, then we can easily define a similar structure on $G^{(2)}$. Indeed, the norm on $G^{(2)}$ can be taken as

$$\|(x,y)\|^{(2)} = \max\left\{\|x\|, \|y\|\right\},$$

and dilatations

$$\delta_\varepsilon^{(2)}(x,y) = (\delta_\varepsilon x, \delta_\varepsilon y).$$

We leave to the reader to check that $G^{(2)}$ endowed with this norm and these dilatations is indeed a normed group with dilatations.

Theorem 16. *Let $(G, \delta, \|\cdot\|)$ be a locally compact normed group with dilatations and $(G^{(2)}, \delta^{(2)}, \|\cdot\|^{(2)})$ be the associated normed group with dilatation. Then the operation (op function) is differentiable.*

Proof. Indeed, we start from the formula (easy to check in $G^{(2)}$):

$$(x,y)^{-1} = (x^{-1}, xy^{-1}x^{-1}).$$

We have then

$$\delta_\varepsilon^{(x,y)}(u,v) = (x\delta_\varepsilon(x^{-1}u), \left(\delta_\varepsilon(x^{-1}u)\right)^{-1} y\delta_\varepsilon(x^{-1}u)\delta_\varepsilon\left(u^{-1}xy^{-1}x^{-1}uv\right)).$$

Let us define

$$Q^{(x,y)}(u,v) = op(x,y)\beta((x,y)^{-1}(u,v)).$$

We have then

$$\frac{1}{\varepsilon} d\left(op\left(\delta^{(x,y)}(u,v)\right), \delta^{op(x,y)} Q^{(x,y)}(u,v)\right) =$$
$$= \frac{1}{\varepsilon} d\left(\delta_\varepsilon \beta_\varepsilon((x,y)^{-1}(u,v)), \delta_\varepsilon \beta((x,y)^{-1}(u,v))\right).$$

The right hand side of this equality converges then to 0 as $\varepsilon \to 0$. More precisely, we have

$$\sup\left\{\frac{1}{\varepsilon} d\left(op\left(\delta^{(x,y)}(u,v)\right), \delta^{op(x,y)} Q^{(x,y)}(u,v)\right) \ : \ d^{(2)}((x,y),(u,v)) \leq \varepsilon\right\} =$$
$$= \sup\left\{d^e\left(\beta_\varepsilon((x,y)^{-1}(u,v)), \beta((x,y)^{-1}(u,v))\right) \ : \ d^{(2)}((x,y),(u,v)) \leq \varepsilon\right\} + \mathcal{O}(\varepsilon).$$

The proof is done. □

In particular we have $Q^{(e,e)}(u,v) = \beta(u,v)$, which shows that the operation β is the differential of the operation *op* computed in the neutral element of $G^{(2)}$.

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